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ORBITS OF THE PSEUDOCIRCLE

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ABSTRACT. The following theorem is proved.

THEOREM. *The pseudocircle has uncountably many orbits under the action of its homeomorphism group. Each orbit is the union of uncountably many composants.*

A pseudocircle is a circularly chainable, hereditarily indecomposable, separating plane continuum. R. H. Bing [**1**] constructed a pseudocircle in 1951 and asked two questions. Are any two pseudocircles homeomorphic? Is a pseudocircle homogeneous?

In 1968, L. Fearnley [**6**] proved that the answer to the first question is yes, and Fearnley [**5**] and J. T. Rogers, Jr. [**14**] independently proved that the answer to the second question is no. Since the advent of the Effros theorem, several elegant proofs of the nonhomogeneity of the pseudocircle have appeared [**7**, **10**, and **13**].

In 1968, after seeing the proof that the pseudocircle is not homogeneous, F. B. Jones asked the second author how many orbits the pseudocircle had under the action of its homeomorphism group, but the matter was not pursued further. Recently, however, the question has arisen again, and Wayne Lewis [**3**] has asked if the pseudocircle has uncountably many orbits.

The purpose of this paper is to prove the following theorem, which gives an affirmative answer to this question.

THEOREM. *The pseudocircle has uncountably many orbits under the action of its homeomorphism group. Each of these orbits is the union of uncountably many composants.*

An interesting sidelight of the proof is the construction of two homeomorphic sets, one being an open set of the pseudocircle and the other being an open set of the pseudoarc.

During the proof we construct an uncountable, abelian subgroup of the homeomorphism group of the pseudocircle. The homeomorphism constructed by Handel [**8**] might be an element of this group.

Each orbit of the pseudocircle is a Borel set [**4**]. Lewis has announced that no orbit can be a G_δ. In the course of our argument we prove this, but we do not determine further restrictions on the type of the Borel set for the orbit.

Question. What classes of Borel sets occur as the orbits of the pseudocircle? In particular, are each two orbits of the same class?

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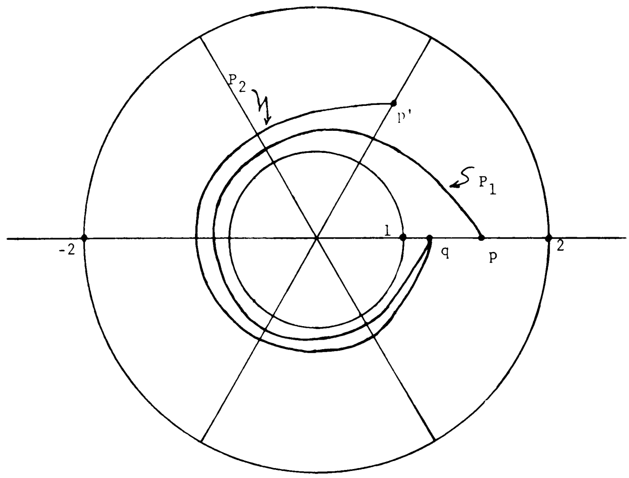

FIGURE 1

A continuum is *circle-like* or *circularly chainable* if it has covers of arbitrarily small mesh whose nerves are circles. An *arc-like* or *chainable* continuum is defined similarly.

A *pseudoarc* is an arc-like, hereditarily indecomposable continuum. Each proper nondegenerate subcontinuum of a pseudocircle is a pseudoarc. The pseudoarc has the fixed-point property [**8**].

If X is a compactum, then $H(X)$ denotes its homeomorphism group with the compact-open topolgy. The *orbit* of a point x in X is $\{y\colon h(y) = x$, for some h in $H(X)\}$.

If A is a collection of sets, then A^* denotes the union of the sets in A. If B is another collection of sets, then B is an *amalgamation* of A if $A^* = B^*$ and each set in B is the union of some sets in A. If the closure of each set in A is a subset of a set in B, then A is said to *closure-refine* B. If A is a chain cover of a continuum X, we shall assume each link of A contains a point not in the closure of any other link of A. Such a chain is said to cover X *essentially.*

1. Embedding an annulus in an annulus. The key idea of this section is to describe a wiggly embedding of an annulus into a standard annulus so that the embedded annulus is mapped onto itself by rotation through an angle of $\frac{\pi}{3}$. Afterwards, we hint at how to continue the process. Emphasis is on intuition; more rigor and detail is provided in the next section.

We use polar coordinates in the plane. We speak of chains with links being closed 2-cells that intersect only in an arc in their boundaries. One can get open sets as links by fattening these slightly.

Consider the closed, planar annulus A bounded by the circles $r = 1$ and $r = 2$. Divide A into 6 congruent parts invariant under rotation by $\pi/3$, as indicated in Figure 1. Let P_1 be an arc in A winding once around the origin in the counterclockwise direction, with initial point $p = (1\frac{3}{4}, 0)$ and terminal point $q = (1\frac{1}{4}, 0)$ and with r-coordinate decreasing monotonically. Let P_2 be a similar arc from $p' = (1\frac{3}{4}, \frac{\pi}{3})$ to q. These arcs are pictured in Figure 1.

Adjust P_1 and P_2 if necessary so that they intersect only in point q. Let $P = P_1 \cup P_2$. Let h be the rotation of the plane by $\frac{\pi}{3}$. Note that $h(p) = p'$. Adjust P if necessary so that $P \cap h(P) = \{p'\}$.

The union of the six arcs $h^n(P)$, $n = 0, 1, \ldots, 5$, is a simple closed curve C_2 in A invariant under h. Thicken C_2 to a neighborhood A_2 homeomorphic to an annulus, so that A_2 is also invariant under h.

Let B_2 be the piece of A_2 that forms a thickening of P so that the boundary of B_2 *in* A_2 consists of a short line segment L_1 intersecting $\theta = 0$ and a line segment L_2 intersecting $\theta = \frac{\pi}{3}$ congruent to L_1 by the rotation of $\frac{\pi}{3}$. Divide B_2 with "radial" lines into links. Use h to rotate B_2 around the annulus A so that the links of B_2 determine $6n$ links in A_2, where n is some integer.

Consider a "rotation" of A_2 defined by moving each link homeomorphically onto its successor link. There are $6n$ of these "rotations," including the identity. Note that the "rotations" corresponding to $0, n, 2n, 3n, 4n$, and $5n$ agree with rotations of the plane through multiples of $\frac{\pi}{3}$ (This is the heart of the construction.).

To continue, embed appropriately a simple closed curve C_3 in the annular neighborhood A_2 so that C_3 is invariant under the $6n$ "rotations" of A_2. The curve C_3 will be the union of $6n$ arcs, each beginning in some link of A_2 and ending in its successor link. Thicken C_3 to A_3, also invariant under these "rotations." Chop A_3 into links and continue.

2. An uncountable abelian group of homeomorphisms of the pseudocircle. The pseudocircle will be defined as the intersection of annuli, so we fill in some details and provide some rigor to the intuitive description of the previous section.

The first modification is that the arcs P_1 and P_2 of the previous section must be embedded crookedly in A with respect to the six links of A. Hence we lose the "r-coordinate decreasing monotonically." This was only used, however, as an aid to intuition. The arcs P_1, P_2, and P are still to enjoy all the properties of paragraph four of the previous section.

The second adjustment is that the simple closed curve C_2 is to be smoothly embedded. Choose the neighborhood A_2 of C_2 so that it is also smoothly embedded and thin.

Hence we have a smoothly embedded annulus A_2 and a smoothly embedded simple closed curve C_2 that are both invariant under the rotation h. The next step is to describe precisely the "links" and "rotations" of A_2.

Let C_2' be the circle in the plane centered at the origin such that the circumference of C_2' is equal to the length of C_2. Let $f_2 \colon C_2 \to C_2'$ be a diffeomorphism that preserves length. Extend f_2 to a homeomorphism (also called f_2) taking A_2 onto an annulus A_2' about C_2'. Require f_2 to map L_1 onto a line segment L_1' on $\theta = 0$ and to map L_2 onto a line segment L_2' on $\theta = \frac{\pi}{3}$. Note that $hf_2 = f_2h$ on A_2.

Let $B_2' = f_2(B_2)$. Use straight lines through the origin to divide B_2' into a number n_2 of congruent pieces. With these pieces as links, B_2' is a chain. The inverses under f_2 of these links are the links of B_2. The number n_2 should be chosen so that (as chains with open sets as links) B_2 is crooked in A, any three consecutive links of B_2 lie in a link of A, and each link of B_2 has small diameter.

By rotation of $\frac{\pi}{3}$, divide all of A_2', and hence A_2, into links. Each of A_2 and A_2' will then have $6n_2$ links, and A_2 is crooked in A. Let h_2' be the rotation of the plane that takes each link of A_2' onto its successor; formally, h_2' is the rotation through the angle $2\pi/6n_2$. Note that $(h_2')^{n_2} = h$.

Let $h_2 = (f_2)^{-1} \circ h_2' \circ f_2$ be defined on A_2. The homeomorphism h_2 is a "rotation" of A_2. It moves each link of A_2 to its successor link. The crux of the matter again is that $(h_2)^{n_2} = h$.

To construct the pseudocircle by induction, it is sufficient to describe a certain embedding of an annulus in A_2. Embed a simple closed curve C_3 into the annulus A_2' in the same manner that C_2 was embedded in A. The curve C_3 will be the union of $6n_2$ arcs, each beginning in some link of A_2 and ending in its successor link. Each of these arcs will be the union of two crooked arcs, one of which winds once around the origin in a counterclockwise direction, as did P_1, and the other returning to the adjacent link, as did P_2. No two of these $6n_2$ arcs will intersect, except possibly in an endpoint. The rotation h_2' will permute these arcs, so that C_3 is invariant under h_2'.

The annular neighborhood A_3 of C_3 will be homeomorphic by f_3 to an annulus A_3' and will have $6n_2n_3$ links, appropriately selected. The rotation h_3' is defined as rotation through the angle $2\pi/6n_2n_3$. It has the property that $(h_3')^{n_3} = h_2'$.

The inverse of f_2 takes A_3 onto an embedded annulus in A (also called A_3). The homeomorphism $h_3 = (f_2 \circ f_3)^{-1} \circ h_3' \circ (f_2 \circ f_3)$ will be a "rotation" of A_3, and $(h_3)^{n_3} = h_2$.

The pseudocircle X is defined as $\bigcap_{n=1}^{\infty} A_n$, where $A_1 = A$ and A_n is defined above.

THEOREM 1. *There exists an uncountable abelian group G of homeomorphisms of the pseudocircle X.*

PROOF. Let $h_1 = h, h_2, h_3, \ldots$ be the "rotations" (described above) constructed with the pseudocircle X. Each h_n generates a finite cyclic group R_n of $H(X)$; furthermore, there are inclusions $R_1 \subset R_2 \subset \cdots$. The direct limit of $\{R_n\}$ yields a countable abelian subgroup of $H(X)$ containing each R_n. Let G be the closure of this group in $H(X)$; then G is also an abelian group. Furthermore, since the identity map of $X = \lim_{n\to\infty} h_n$ and since G is homogeneous, it follows that G is perfect and hence uncountable.

A continuum Z is said to be *nearly homogeneous* if given a point z in Z and a nomempty open set 0 of Z, there exists a homeomorphism of Z onto itself taking z into 0.

THEOREM 2. *The pseudocircle is nearly homogeneous.*

PROOF. The orbit of a point under G is dense in X.

THEOREM 3. *If $h: X \to X$ is a homeomorphism of the pseudocircle and L is a composant of X such that $h(L) = L$, then h has a fixed point in L.*

PROOF. Let x be some point of L. Since x and $h(x)$ belong to L, there exists an irreducible continuum Z containing x and $h(x)$. The continuum Z is either a point or a pseudoarc. Since the continuum $Z \cup h(Z)$ is indecomposable, either $h(Z) \subset Z$ or $Z \subset h(Z)$. Assume $h(Z) \subset Z$ (if not, replace h with h^{-1}). Let $N = \bigcap_{n=1}^{\infty} h^n(Z)$. Hence N is a subcontinuum of Z satisfying $h(N) = N$. Since N is either a point or a pseudoarc, there exists a point in N fixed by h.

THEOREM 4. *The orbit of a point of X under G is a dense subset of X intersecting uncountably many composants of X.*

PROOF. We show that uncountably many members of G do not map a fixed composant of X to the same composant.

Consider the uncountable collection $K = \{g \in G : g = \lim g_i$, where $g_i = h_1^{K_1} \circ h_2^{K_2} \circ \cdots \circ h_i^{K_i}$ and $K_i = 0$ or $2\}$. Since any three consecutive (open) links of A_n are contained in an (open) link of A_{n-1}, each $\lim g_i$ does exist. Furthermore, the only homeomorphism of K that fixes a point is the identity. In fact, no two homeomorphisms of K map a fixed composant of X to the same composant.

3. Almost chainable continua without a dense G_δ-orbit. A continuum is *almost chainable* if, for each positive ε, there exists an ε-covering $\mathcal{D}$ of X and a chain $\mathcal{C} = \{C_1, \ldots, C_n\}$ of elements of $\mathcal{D}$ such that no C_i $(1 < i \leq n)$ intersects an element of $\mathcal{D} - \mathcal{C}$ and every point of X is within ε of some point of $\mathcal{C}^*$. The set C_n is called an *end link* of $\mathcal{C}$.

C. E. Burgess introduced this fruitful notion in [**2**] and proved that the pseudocircle is almost chainable.

In this section we give sufficient conditions for an almost chainable continuum to fail to have a dense G_δ-orbit. The pseudocircle satisfies these conditions by [**2**, **14**], and the construction of §1, and so it does not have a dense G_δ-orbit.

Our theorem generalizes a theorem of Hagopian [**7**], who used almost chainability to provide another proof that the pseudocircle is not homogeneous.

A continuum X is said to have property K at a point a of X provided that, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if b is in X with $d(a,b) < \delta$ and A is a subcontinuum of X containing a, then there is a subcontinuum B of X containing b such that the Hausdorff distance from A to B is less than ε. A continuum has property K if it has property K at each point.

THEOREM 5. *Suppose X is an almost chainable continuum that has property K. If X has a homeomorphism without any fixed point, then X does not have a dense G_δ-orbit.*

PROOF. Let $f: X \to X$ be a homeomorphism without a fixed point. Let ε be a positive number such that $d(f(x), x) > 2\varepsilon$ and $d(f^{-1}(x), x) > 2\varepsilon$.

Suppose the theorem is not true, so that X has a dense G_δ-orbit A. We can apply the Effros theorem [**4**] to obtain a countable collection $\mathcal{W}$ of open sets of X such that (1) $A \subset \mathcal{W}^*$; and (2) $\mathcal{W}$ has the ε-push property with respect to A, that is, if $W \in \mathcal{W}$ and $x, y \in A \cap W$, then there is some homeomorphism h in $H(X)$ such that $h(x) = y$ and h moves no point of X more than ε.

For each j in N, there exists a $\frac{1}{j}$-cover $\mathcal{D}_j$ of X and a chain $\mathcal{C}_j = \{C(j,i) \mid 1 \leq i \leq n_j\}$ of elements of $\mathcal{D}_j$ such that no element of $\mathcal{D}_j - \mathcal{C}_j$ intersects $\{C(j,i) \mid 2 \leq i \leq n_j\}^*$ and every point of X is within $\frac{1}{j}$ of some element of $\mathcal{C}_j^*$.

For each j in N, choose a point p_j in $C(j, n_j)$, and let K_j denote the p_j-component of $X - C(j,1)$. Without loss of generality assume that $p_1, p_2, \ldots$ converges to the point p of X, and note that the sequence $K_1, K_2, \ldots$ of continua converges to X.

Choose a point x in $A - \{p\}$, and then choose a sequence $x_1, x_2, \ldots$ of points of K_i that converges to x. For each i there is some r_i such that $x_i \in C(i, r_i)$. Use the fact that X has property K at x_i to find a continuum $\hat{K}_i$ such that (1) $\hat{K}_i$ is within $\frac{1}{i}$ of K_i with respect to the Hausdorff metric, (2) there is a point $\hat{x}_i$ in $A \cap \hat{K}_i \cap C(i, r_i)$, (3) $\hat{K}_i \subset \mathcal{C}_i^*$, and (4) $\hat{K}_i \cap C(i,j) \neq \varnothing$, for $1 < j \leq n_i$.

Choose $\hat{p}_i \in \hat{K}_i \cap C(i, n_i)$. Note that $\hat{x}_i, \hat{x}_2, \dots$ converges to x, $\hat{p}_1, \hat{p}_2, \dots$ converges to p, and $\hat{K}_1, \hat{K}_2, \dots$ converges to X. There is some open set U such that $x \in U$, $U \cap f(U) = \varnothing$, and both U and $f(U)$ are subsets of some sets W and W' in $\mathcal{W}$. There is some j in N such that (1) x_j and $\hat{x}_j$ are in U, (2) $C(j, r_j) \subset U$, (3) $\frac{1}{j} < \varepsilon$, and (4) there is an m_j such that $C(j, m_j) \subset f(U)$. For simplicity, let $r_j = n$, $m_j = m$.

Either $n < m$ or $m < n$. Suppose first that $n < m$. Since $f(\hat{x}_j) \in f(U)$, there is a homeomorphism h in $H(X)$ such that $hf(\hat{x}_j) \in C(j, m)$ and h moves no point of X more than ε. Note that hf moves each point of X a distance greater than ε.

For $1 \leq k \leq n_j$, let

$$A_k = \{y \in \hat{K}_j \cap C(j,k) | hf(y) \in \{C(j,i) | k \leq i \leq n\}^*\}$$

and

$$B_k = \{y \in \hat{K}_j \cap C(j,k) | hf(y) \notin \{C(j,i) | k \leq i \leq n\}^*\}.$$

Let $A = \bigcup\{A_k : k = 1, \dots, n_j\}$ and $B = \bigcup\{B_k : k = 1, \dots, n_j\}$. Then $\hat{x}_j \in A$ and $\hat{p}_j \in B$, $A \cup B = \hat{K}_j$, and A and B are mutually exclusive closed point sets. Since this is a contradiction, it must be the case that $m < n$.

Since $C(j, m) \cap \hat{K}_j \neq \varnothing$, we can use property K again to obtain a continuum L_j of X such that (1) $L_j \subset \mathcal{C}_j^*$, (2) $L_j \cap C(j, m) \cap A \neq \varnothing$, and (3) L_j intersects each link of $\mathcal{C}_j - \{C(j, 1)\}$. Choose $q_j \in L_j \cap C(j, n_j)$. Suppose $z \in L_j \cap C(j, m) \cap A$. There is h' in $H(X)$ such that $h'f^{-1}(z) \in C(j, n)$ and h' moves no point of X more than ε.

For $1 \leq k \leq n_j$, let

$$A'_k = \{y \in L_j \cap C(j,k) | h'f^{-1}(y) \in \{C(j,i) | k \leq i \leq n\}^*\},$$

and let

$$B'_k = \{y \in L_j \cap C(j,k) | h'f^{-1}(y) \notin \{C(j,i) | k \leq i \leq n\}^*\}.$$

Let $A' = \bigcup A'_k$ and $B = \bigcup B'_k$. Then $z \in A'$, $q_j \in L_j \cap C(j, n_j) \subset B'$, and A' and B' are disjoint closed point sets whose union is L_j. Again we have a contradiction, and it follows that X has no dense G_δ-orbit.

4. Homeomorphic open sets of the pseudoarc and the pseudocircle. A cover $\mathcal{U}$ of a compactum is said to be *taut* if $\overline{U} \cap \overline{U}' \neq \varnothing$, U and $U' \in \mathcal{U}$, implies $U \cap U' \neq \varnothing$.

The notion of a pattern, defined next, is used to describe the way one chain is embedded in another. If A and B are subsets of the nonnegative integers, then a function $f: A \to B$ is a *pattern* if whenever $i, i+1 \in A$, then $|f(i+1) - f(i)| \leq 1$. If $\mathcal{V} = \{V_i | i \in A\}$ and $\mathcal{U} = \{U_i | i \in B\}$ are collections of open sets of the compactum X, then $\mathcal{V}$ follows the pattern f in $\mathcal{U}$ provided $V_i \subseteq U_{f(i)}$ for each $i \in A$. We will call f a pattern on $\mathcal{U}$.

We also must describe the way one finite collection of chains is embedded in another such collection. To do this, we must speak of a pattern that a collection of chains follows in another collection of chains. In such a pattern, there are two subscripts; the first indicates a chain and the second indicates a link of that chain. Here is the definition of such a pattern: If A and B are two subsets of $(N \cup \{0\}) \times (N \cup \{0\})$, then the function $f: A \to B$ is said to be a *pattern* provided that (1) whenever $(i,k), (i,k') \in A$, $f(i,k)_1 = f(i,k')_1$ (the first coordinates are the same); and (2) whenever $(i,k), (i,k+1) \in A$, $|f(i,k+1)_2 - f(i,k)_2| \leq 1$ (the second coordinates differ by no more than 1).

If $\mathcal{V} = \{V_{(i,j)} | (i,j) \in A\}$ and $\mathcal{U} = \{U_{(i,j)} | (i,j) \in B\}$ are collections of open sets of the compactum X, then $\mathcal{V}$ follows the pattern f in $\mathcal{U}$ provided $V_{(i,j)} \subseteq U_{f(i,j)}$ for each $(i,j) \in A$. Again f is a pattern on $\mathcal{U}$. We write V_{ij} for $V_{(i,j)}$.

THEOREM 6. *Let $U_1, U_2, \ldots$ and $V_1, V_2, \ldots$ be sequences of finite collections of chains in the compact metric spaces X and Y, respectively, such that for each i*

(1) *U_i and V_i consist of a finite collection of taut chains such that if C and C' are two different chains in U_i, then $C^* \cap C'^* = \varnothing$ in X, and if D and D' are two different chains in V_i, then $D^* \cap D'^* = \varnothing$;*

(2) $U_i^{**} = X$, $V_i^{**} = Y$;

(3) $\lim_i \operatorname{mesh} U_i = \lim_i \operatorname{mesh} V_i = 0$;

(4) *the number of chains in U_i is the same as the number of chains in V_i and if $U(i,1), \ldots, U(i,n_i)$ is a listing of the chains in U_i, we can list the chains in $V_i = V(i,1), \ldots, V(i,n_i)$ so that for each j, $|U(i,j)| = |V(i,j)|$;*

(5) *both U_{i+1} and V_{i+1} follow a pattern f_i in U_i and V_i, respectively, with respect to the listings in* (4);

(6) *each link of each chain of U_i contains a point of X not contained in any other link of a chain U_i (and similarly for V_i and Y).*

Then there exists a homeomorphism h taking X onto Y.

PROOF. For each j, $1 \leq j \leq n_i$, denote the chain $U(i,j)$ by $\{U(i,j,0), \ldots, U(i,j,a_{i,j})\}$ and the chain $V(i,j)$ by $\{V(i,j,0), \ldots, V(i,j,a_{i,j})\}$. For each i, define $h_i: U_i^{**} \to V_i^{**}$ by $h_i: U(i,j,k) = V(i,j,k)$ for $1 \leq j \leq n_i$, $0 \leq k \leq a_{ij}$. Note that h_i is one-to-one and onto, and that h_i takes adjacent links of chains of U_i to adjacent links of chains of V_i.

From f_i define $\hat{f}_i: U_{i+1} \to U_i$ and $f_i': V_{i+1} \to V_i$ by $\hat{f}_i(U(i+1,j,k)) = U(i,l,m)$ if $f_i(j,k) = (l,m)$ and $f_i'(V(i+1,j,k)) = V(i,l,m)$ if $f_i(j,k) = (l,m)$.

For each x in X, there is a sequence $C(1,x)$, $C(2,x), \ldots$ such that (1) for each i, $x \in C(i,x)$ and $C(i,x) \in U_i^*$, and (2) $\hat{f}_i(C(i+1,x)) = C(i,x)$. Then $\{x\} = \bigcap_i C(i,x)$ and $\bigcap_i \overline{h_i C(i,x)}$ is nonempty and degenerate. Note that $f_{i-1}' h_i(C(i,x)) = h_{i-1}\hat{f}_{i-1}(C(i,x))$. Then define $h(x) = \bigcap_i \overline{h_i(C(i,x))}$. We shall prove h is a homeomorphism.

The function h is injective. Let x and x' be points of X. There exists an integer i such that if L is a link of a chain of U_i containing x, and L' is a link of a chain of U_i containing x', then the closures of L and L' are disjoint. Hence the closures of $h_i(L)$ and $h_i(L')$ are disjoint. Therefore $h(x) \neq h(x')$.

The function h is continuous. If y is a point of $h(X)$, then $y = \bigcap_i \overline{h_i C(i,x)}$ for some x in X. Let O be an open set in Y containing y. We will show that there exists an integer i such that $h_i(C(i,x)) \subset O$.

Choose i so large that the closure of $h(C(i,x))$ as well as the closure of any link adjacent to $h_i(C(i,x))$ is contained in O. If $z \in C(i,x)$, then $C(i,z) = C(i,x)$ or a link adjacent to $C(i,x)$. In either case $\overline{h_i(C(i,z))} \subset O$. Hence $h(z) \in O$.

The function h is surjective. It suffices to show $h(X)$ is dense in Y. Suppose there exists a link K of a chain of V_i such that $\mathrm{Cl}(K) \cap h(X) = \emptyset$. Let L be the link of the chain U_i such that $h_i(L) = K$. Condition (6) of the hypothesis implies that L contains a point x such that $C(i,x) = L$. Hence $h_i(C(i,x)) = K$. Therefore $h(x) \in \mathrm{Cl}(K)$. This is a contradiction.

THEOREM 7. *Suppose X is the pseudoarc and Y is the pseudocircle. Then there are open sets O of X and U of Y and a homeomorphism $h: \overline{O} \to \overline{U}$ such that $h(\partial O) = \partial U$.*

PROOF. Suppose that $\mathcal{O} = \mathcal{O}_1, \mathcal{O}_2, \ldots$ is a defining sequence of taut chain covers of X such that for each i (1) mesh $\mathcal{O}_i < 2^{-i}$, and (2) $\mathcal{O}_i$ is an amalgamation of $\mathcal{O}_{i+1}$. Suppose that $\mathcal{U} = \mathcal{U}_1, \mathcal{U}_2, \ldots$ is a defining sequence of taut circular chain covers of Y such that for each i, (1) mesh $\mathcal{U}_i < 1/2^i$, and (2) $\mathcal{U}_i$ is an amalgamation of $\mathcal{U}_{i+1}$. Without loss of generality, assume that $\mathcal{O}_1$ and $\mathcal{U}_1$ both have at least three links.

Now choose $O \in \mathcal{O}_1$ such that O is not an end link, and choose $U \in \mathcal{U}_1$. For each i, let $\mathcal{C}_i = \{O' \in \mathcal{O}_{i+1} | O' \subseteq O\}$ and let $\mathcal{D}_i = \{U' \in \mathcal{U}_{i+1} | U' \subseteq U\}$. Note that $\mathcal{C}_i^* = O$ and $\mathcal{D}_i^* = U$.

It is the case that if $x \in \overline{O} - O$ and $i \in N$, then x is in the closure of exactly one link C of $\mathcal{C}_i$ because if $x \in \overline{C} \cap \overline{C}'$, where both C and C' are in $\mathcal{C}_i$, then $x \in C \cup C'$ (tautness) and so $x \in O$.

Thus for each i, $\mathcal{C}_i' = \{C \cup (\overline{C} \cap (\overline{O} - O)) | C \in \mathcal{C}_i\}$ is an open cover of $\overline{O}$ consisting of a finite collection of maximal chains. For each i, let $C(i)$ denote the set of all maximal chains in $\mathcal{C}_i'$ and let $D(i)$ denote the set of all maximal chains in $\mathcal{D}_i' = \{D \cup (\overline{D} \cap (\overline{U} - U)) | D \in \mathcal{D}_i\}$. Then list the chains in each of $C(i)$ and $D(i)$: $C(i) = \{C(i,1), \ldots, C(i,n_i)\}$ and $D(i) = \{D(i,1), \ldots, D(i,m_i)\}$; and list the links in each chain (in order): for $1 \le j \le n_i$, $C(i,j) = \{C(i,j,0), \ldots, C(i,j,a_{i,j})\}$, and for $1 \le j \le m_i$, $D(i,j) = \{D(i,j,0), \ldots, D(i,j,b_{i,j})\}$. Note that

(1) $C(i,j)^* \cap C(i,j')^* = \emptyset$ if $j \neq j'$;

(2) $C(i,j)^*$ is open and closed in $\overline{O}$;

(3) $C(i,j)^*$ contains a continuum Z_{ij} such that $Z_{ij} \cap (C(i,j,0) - O) \neq \emptyset$,

$$Z_{ij} \cap (C(i,j,a_{i,j}) - O) \neq \emptyset, \quad \text{and } Z_{ij} \subseteq \overline{O}.$$

Analogous properties hold for the $D(i,j)$'s.

The idea of the proof is the following: We will construct a sequence $\hat{C}(1), \hat{C}(2), \ldots$ of open covers of $\overline{O}$ such that each $\hat{C}(i)$ consists of a finite collection of taut maximal chains, and a sequence $\hat{D}(1), \hat{D}(2), \ldots$ of open covers of $\overline{U}$ such that each $\hat{D}(i)$ consists of a finite collection of taut maximal chains. Further, for each i there will be a pattern f_i such that $\hat{C}(i+1)$ follows f_i in $\hat{C}(i)$, and $\hat{D}(i+1)$ follows f_i in $\hat{D}(i)$. We will also make sure that the hypotheses of the previous theorem are satisfied by the sequences $\hat{C}(1), \hat{C}(2), \ldots$ and $\hat{D}(1), \hat{D}(2), \ldots$ and then we will apply this theorem to conclude that $\overline{O}$ is homeomorphic to $\overline{U}$.

Now define $\hat{C}(1)$ to be $C(1)$: so for $1 \le j \le n_1$ and $0 \le k \le a_{1,j}$, $\hat{C}(1,j,k) = C(1,j,k)$ and $\hat{C}(1,j) = C(1,j)$. Let $N_1 = n_1$ and $A_{1,j} = a_{1,j}$.

There is an integer k_1 such that $D(k_1)$ consists of at least N_1 chains and, for some N_1 chains in $D(k_1)$, each of these chains has at least M links, where $M = \max\{|\hat{C}(i,j)| : j \le N_1\}$. Assume without loss of generality that these N_1 chains are listed first in $D(k_1)$.

In the next three paragraphs, we define $\hat{D}(1)$. The first link of the first chain of $\hat{D}(1)$ is the first link of the first chain of $D(k_1)$. Formally $\hat{D}(1,1,1) = D(k_1,1,1)$. Similarly, the second link $\hat{D}(1,1,2)$ is defined to be $D(k_1,1,2)$. A similar definition is used until we get to the last link $\hat{D}(1,1,A_{1,1})$ of the first chain of $\hat{D}(1)$, which is defined to be the union of $D(k_1,1,A_{1,1})$ and any other links of the first chain of $\hat{D}(1)$ that follow it.

The links of the second chain of $\hat{D}(1)$ are defined similarly. In fact, a similar definition is used until we get to the last link of the last chain of $\hat{D}(1)$: this link is $\hat{D}(1,N_1,A_{1,N_1})$ and is defined as the union of $D(k_1,N_1,A_{1,N_1})$ and all the links of $D(k_1,N_1)$ that follow it and all the links of all the chains of $D(k_1)$ that have not yet been considered.

Define $\hat{D}(1) = \{\hat{D}(1,1), \hat{D}(1,2), \dots, \hat{D}(1,N_1)\}$, where $\hat{D}(1,i)$ is the ith chain of $\hat{D}(1)$.

There is some $k_2 > k_1$ such that $D(k_2)$ closure-refines $\hat{D}(1)$, $\operatorname{mesh} D(k_2) \le \frac{1}{4}$, and 2^{-k_2-1} is less than the distance between ∂U and

$$\overline{\{\hat{D}(1,i,j) | 1 \le i \le N_1,\ 0 < j < A_{1,1}\}^*}.$$

Define $\hat{D}(2) = D(k_2)$. Hence, for $1 \le j \le m_{k_2} \equiv M_2$ and $0 \le l \le b_{k_2 j} \equiv B_{2j}$, we have $\hat{D}(2,j,l) = D(k_2,j,l)$.

Now $\hat{D}(2)$ follows some pattern f_1 in $\hat{D}(1)$ so that $f_1 \colon \alpha \to \beta$, where

$$\alpha = \{(i,j) | 1 \le i \le M_2,\ 0 \le j \le B_{2i}\},$$
$$\beta = \{(i',j') | 1 \le i' \le N_1,\ 0 \le j' \le A_{1i'}\}.$$

We must construct $\hat{C}(2)$ so that it follows f_1 in $\hat{C}(1)$. To this end, for $1 \le i \le M_2$, let

$$F_i = \{\hat{D}(i,j,l) |\text{ there is } k \text{ such that } (i,k) \text{ is in } \alpha \text{ and } f_1(i,k) = (j,l)\}.$$

Thus F_i is a subchain of $\hat{D}(1,j)$ that contains either the first link or the last link of $\hat{D}(1,j)$.

There is a sequence $G_1, G_2, \dots, G_{M_2}$ of mutually exclusive open and closed point sets of $\overline{O}$ such that

(4) $\overline{O} = G_1 \cup G_2 \cup \cdots \cup G_{M_2}$;

(5) there is l in N such that each G_i is a union of some of the links in $C(l)^*$;

(6) if, for each i, $H_i = \{\hat{C}(1,j,k) \in \hat{C}(1,j) | \hat{D}(1,j,k) \in F_i\}$ (where $G_i \subset \hat{C}(1,j)^*$), then $G_i \subseteq H_i^*$; and

(7) H_i essentially covers G_i.

To see that such a sequence of open sets exists, consider the following: There is a pseudoarc P, a proper subcontinuum of X, such that

(8) some subcontinuum P' of P is a subset of $\overline{O}$,

(9) $P \cap \overline{O} \subseteq \hat{C}(1,1)^*$,

(10) $\hat{C}(1,1,0) \cap P'$ contains a limit point of $P - \overline{O}$, and

(11) $\hat{C}(1,1,A_{1,1}) \cap P'$ contains a limit point of $P - \overline{O}$.

Without loss of generality, suppose that $\hat{D}(2,1), \dots, \hat{D}(2,n)$ is the list of all the chains D in $\hat{D}(2)$ such that $D^* \subseteq \hat{D}(1,1)^*$, and that associated with $\hat{D}(2,1)$ is a continuum Z that "runs all the way through $\hat{D}(1,1)^*$," i.e., there is some subcontinuum Z' of Z such that $Z' \subseteq \hat{D}(2,1)^*$ and $Z' \cap (\overline{U} - U) \cap \hat{D}(1,1,0) \neq \varnothing$ and $Z' \cap (\overline{U} - U) \cap \hat{D}(1,1,A_{1,1}) \neq \varnothing$.

Choose n mutually exclusive pseudoarcs $P_1, \ldots, P_n$ contained in P such that for each i

(12) some subcontinuum P'_i of P_i is a subset of $\hat{C}(1,1)^*$,

(13) $P_i \not\subset \hat{C}(1,1)^*$ (this will be redundant),

(14) $\hat{C}(1,1,0) \cap P'_i$ contains a limit point of $P - \overline{O}$, and

(15) $\hat{C}(1,1,A_{1,1}) \cap P'_i$ contains a limit point of $P - \overline{O}$.

There is an integer k such that $2^{-k} < \frac{1}{2} \min\{d(P_i, P_j) | i \neq j\}$. For each i there is an integer $k'_i > k$ such that

(16) some chain E_i of $\mathcal{O}_{k'_i}$ essentially covers some subcontinuum of P_i,

(17) neither end link of E_i intersects $\overline{O}$,

(18) $\hat{C}(1,1,0) \cap \partial O$ contains a limit point of $E_i^* - \overline{O}$, and

(19) $\hat{C}(1,1,A_{1,1}) \cap \partial O$ contains a limit point of $E_i^* - \overline{O}$.

Note that $E_i^* \cap E_j^* = \varnothing$ if $i \neq j$.

Use the crookedness of the chains covering X to find a $k_i > k'_i$ such that there is a subchain L_i of $\mathcal{O}_{k_i}$ such that

(20) neither end link of L_i intersects $\overline{O}$,

(21) $L_i^* \cap \overline{O} \subseteq H_i^*$,

(22) each link of H_i contains a link of L_i which does not intersect any other link of H_i,

(23) $L_i^* \subset E_i^*$,

(24) $\hat{D}(2,i)^* \cap \hat{D}(1,1,0) \cap \partial U \neq \varnothing$ if and only if some point of $\hat{C}(1,1,0) \cap \partial O$ is a limit point of $L_i^* - \overline{O}$, and

(25) $\hat{D}(2,i)^* \cap \hat{D}(1,1,A_{1,1}) \cap \partial U \neq \varnothing$ if and only if some point of $\hat{C}(1,1,A_{1,1}) \cap \partial O$ is a limit point of $L_i^* - \overline{O}$.

For $i > 1$, define $G_i = L_i^* \cap \overline{O}$ and let $G_1 = \hat{C}(1,1)^* - \{G_i | i > 1\}^*$.

We have thus obtained $G_1, G_2, \ldots, G_n$, and still need to get the G_i's contained in the other chains of $\hat{C}(1)$. But we can do this in exactly the same way. Choose $l > k_i$, for all i.

Now for each i, $1 \leq i \leq M_2$, denote H_i by $\{\hat{C}(1,j,\alpha_i), \ldots, \hat{C}(1,j,\beta_i)\}$ (where $H_i^* \subseteq \hat{C}(1,j)^*$).

For $1 \leq i \leq M_2$, $\alpha_i \leq k \leq \beta_i$, let $g(i,k) = G_i \cap \hat{C}(1,j,k)$ and let $g_i = \{g(i,\alpha_i), \ldots, g(i,\beta_i)\}$. The chain g_i is an open taut cover of the hereditarily indecomposable compactum G_i. Also each link in g_i is a union of links of $C(l)^*$.

For each i, let $f_{i1} = f_1 | \{(i,j) | 0 \leq j \leq B_{2,i}\}$. Next we need to find an open taut cover V_i of G_i such that only end links of V_i contain boundary points of $\overline{O}$ and such that V_i follows f_{i1} in g_i (because $\hat{D}(2,i)$ follows f_{i1} in $\hat{D}(1,j)$, where $\hat{D}(2,i)^* \subset \hat{D}(1,j)^*$). We indicate how to do this in the most involved case and leave the other cases to the reader.

For convenience, assume $\hat{D}(2,i)^* \subset \hat{D}(1,1)^*$.

Suppose

(26) $g(i,\alpha_i) \cap \partial O \neq \varnothing$, $g(i,\beta_i) \cap \partial O \neq \varnothing$,

(27) $\alpha_i \neq \beta_i$,

(28) $(f_{i1})^{-1}(1,0)$ is not degenerate, and

(29) $(f_{i1})^{-1}(1, A_{1,1})$ is not degenerate.

There are open sets $g'(i,\alpha_i - 1)$ and $g'(i,\alpha_i)$ such that

(30) $\underline{g(i,\alpha_i)} \cap \partial O \subseteq g'(i,\alpha_i - 1)$,

(31) $\overline{g'(i,\alpha_i)} \cap \partial O = \varnothing$,

(32) $g'(i,\alpha_i - 1) \cup g'(i,\alpha_i) = g(i,\alpha_i)$, and

(33) $\overline{g'(i,\alpha_i - 1)} \cap \overline{g(i,k)} = \varnothing$, for all $k > \alpha_i$.

There are open sets $g'(i,\beta_i + 1)$ and $g'(i,\beta_i)$ such that

(34) $\underline{g(i,\beta_i)} \cap \partial O \subseteq g'(i,\beta_i + 1)$,

(35) $\overline{g'(i,\beta_i)} \cap \partial O = \varnothing$,

(36) $g'(i,\beta_i + 1) \cup g'(i,\beta_i) = g(i,\beta_i)$, and

(37) $\overline{g'(i,\beta_i + 1)} \cap \overline{g(i,k)} = \varnothing$, for all $k < \beta_i$.

Then $g'_i = \{g'(i,\alpha_i - 1), g'(i,\alpha_i), g(i,\alpha_i + 1), \ldots, g(i,\beta_i - 1), g'(i,\beta_i), g'(i,\beta_i + 1)\}$ is a taut chain cover of G_i. Furthermore, only $g'(i,\alpha_i - 1)$ and $g'(i,\beta_i + 1)$ contain boundary points of $\overline{O}$. Let $g'(i,j) = g(i,j)$, for $\alpha_i < j < \beta_i$.

Now go to $\hat{D}(1,1)$ and do the same thing. There are open sets $\hat{D}'(1,1,-1)$ and $\hat{D}(1,1,0)$ such that

(38) $\underline{\hat{D}(1,1,0)} \cap \partial U \subseteq \hat{D}'(1,1,-1)$,

(39) $\overline{\hat{D}'(1,1,0)} \cap \partial U = \varnothing$,

(40) $\hat{D}'(1,1,-1) \cup \hat{D}'(1,1,0) = \hat{D}(1,1,0)$,

(41) $\overline{\hat{D}'(1,1,-1)} \cap \overline{\hat{D}(1,1,k)} = \varnothing$, for $k > 0$, and

(42) $\hat{D}'(1,1,-1)$ contains one end link of $\hat{D}(2,i)$ and no other link of $\hat{D}(2,i)$.

There are open sets $\hat{D}'(1,1,A_{1,1})$ and $\hat{D}'(1,1,A_{1,1}+1)$ with similar properties with respect to $\hat{D}(1,1,A_{1,1})$. Let

$$\hat{D}'(1,1) = \{\hat{D}'(1,1,-1), \hat{D}'(1,1,0), \hat{D}(1,1,1), \ldots,\\ \hat{D}(1,1,A_{1,1}-1), \hat{D}'(1,1,A_{1,1}), \hat{D}'(1,1,A_{1,1}+1)\}.$$

Let $\hat{D}'(1,1,j) = \hat{D}(1,1,j)$, for $0 < j < A_{1,1}$.

The chain $\hat{D}(2,i)$ follows a pattern $\hat{f}_{i1}$ in $\hat{D}'(1,1)$, where $\hat{f}_{i1}(i,0)$ is one of $(1,-1)$ or $(1, A_{1,1}+1)$, $\hat{f}_{i1}(i,B_{2i})$ is the other, and $\hat{f}_{i1}(i,j) = f_{i1}(i,j)$, for $0 < j < B_{2i}$. Let us say that $\hat{f}_{i1}(i,0) = (1,-1)$ and $\hat{f}_{i1}(i,B_{2i}) = (1, A_{1,1}+1)$ for convenience.

There exists a taut open chain cover $V_i = \{V(i,0), \ldots, V(i,B_{2i})\}$ of G_i such that V_i follows the pattern $\hat{f}_{i1}$ in g'_i (Theorem 3 of [**12**]). Note that this means end links of V_i are contained in end links of g'_i; in fact, no other link of V_i can contain a boundary point of $\overline{O}$. Moreover, V_i follows the pattern f_{i1} in g_i. Finally, each end link of V_i contains a boundary point of O (additional manipulation of the same sort may be required to achieve this in the additional case).

We are almost ready to define $\hat{C}(2)$. Our only remaining problem is that the links of V_i may not be unions of links of a defining chain.

Let δ be a positive number such that 2δ is less than the distance between the closures of any two nonadjacent links of V_i and also 2δ is less than the distance between $\partial 0$ and the closure of any non-end link of V_i. There is some $l_i > l$ such that $2^{-l_i} < \delta$. For each j, $0 \le j \le B_{2i}$, let $V'(i,j) = \{c \in C'_{l_i} | c \subset g(i, f_{i1}(i,j)_2)$ and $c \cap V(i,j) \neq \varnothing\}^*$, where $f_{i1}(i,j)_2$ denotes the second coordinate of $f_{i1}(i,j)$. Since $g(i, f_{i1}(i,j)_2)$ is a union of links of C'_l and, thus, of C'_{l_i}, it follows that $V(i,j) \subseteq V'(i,j) \subseteq g(i, f_{i1}(k,i,j))$.

Since $2^{-l_i} < \delta$, it follows that $V_i' = \{V'(i,0),\dots,V'(i,B_{2i})\}$ is an open taut chain cover of G_i; V_i' follows f_{i1} in g_i; V_i' is an amalgamation of some links of $\mathcal{C}'_{l_i}$; and only end links of V_i' can contain boundary points of $\overline{O}$.

Define $\hat{C}(2,i,j) = V'(i,j)$, $\hat{C}(2,i) = V_i'$, and $\hat{C}(2) = \{V_1',\dots,V_{M_2}'\}$. Note that $\operatorname{mesh}\hat{C}(2) \leq 1/2$ and $\hat{C}(2)$ follows f_1 in $\hat{C}(1)$.

There is k_3 in N such that $C(k_3)$ closure-refines $\hat{C}(2)$, $\hat{C}(2)$ is an amalgamation of $C(k_3)$, and $\operatorname{mesh} C(k_3) < \frac{1}{8}$. Define, for appropriate j,k, $\hat{C}(3,j,k) = C(k_3,j,k)$; $\hat{C}(3,j) = C(k_3,j)$; $\hat{C}(3) = C(k_3)$. Now obtain $\hat{D}(3)$ in a manner similar to the one in which $\hat{C}(2)$ was just obtained (so that $\hat{D}(3)$ follows a pattern f_2 in $\hat{D}(2)$ that $\hat{C}(3)$ follows in $\hat{C}(2)$), and continue this process. We obtain the desired sequences, and we use the preceding theorem to conclude that there exists a homeomorphism $h\colon \overline{O} \to \overline{U}$.

It remains to show that $h(\partial O) = \partial U$. First we note that $x \in \overline{O} - O$ if and only if $\{x\} = \bigcap c_i$ where for each i, c_i is an end link of a chain in $\hat{C}(i)$. To see this, recall that $x \in \overline{O} - O$ implies that x is in the closure of exactly one link C of one chain of $\hat{C}(i)$, and that link must be an end link of a chain in $\hat{C}(i)$. On the other hand, if x belongs to an end link of some chain in $\hat{C}(i)$ for each i, then $d(x, \overline{O} - O) \leq \operatorname{mesh}\hat{C}(i)$ for all i. Hence $x \in \overline{O} - O$.

Similarly $y \in \overline{U} - U$ if and only if $\{y\} = \bigcap d_i$ where for each i, d_i is an end link of a chain in $\hat{D}(i)$.

Furthermore the patterns f_i take end links to end links, and the homeomorphism h takes a point in $\overline{O}$ that is the intersection of end links to a point in $\overline{U}$ that is an intersection of end links. This completes the proof of the theorem.

THEOREM 8. *If A is an open set of the pseudoarc X such that $X - \overline{A} \neq \varnothing$, then A is homeomorphic to an open set of the pseudocircle. If B is an open set of the pseudocircle Y such that $Y - \overline{B} \neq \varnothing$, then B is homeomorphic to an open set of the pseudoarc.*

PROOF. Without loss of generality, the chain $\mathcal{O}_1 = \{O_1,\dots,O_n\}$ of the previous theorem has the property that neither its first link O_1 nor its last link O_n intersects A.

Let $O = O_2 \cup \cdots \cup O_{n-1}$. Then $\mathcal{O}_1' = \{O_1, O, O_n\}$ is a three-linked chain covering the pseudoarc. Replace $\mathcal{O}_1$ with $\mathcal{O}_1'$ in the preceding proof. It follows that $A \subset O$ and O is homeomorphic to an open set U of the pseudocircle. Hence A is homeomorphic to an open set of the pseudocircle.

A similar proof holds for the second claim.

5. The main theorem. We now have the ingredients for a proof of the following theorem.

THEOREM 9. *The pseudocircle X has uncountably many orbits under the action of its homeomorphism group. Each such orbit is the union of uncountably many composants.*

PROOF. If a continuum Z has only countably many orbits under the action of its homeomorphism group, then one of these orbits is second category in Z and hence second category in itself. It follows from the Effros Theorem [**4**] that Z has a G_δ-orbit. Since the pseudocircle is almost chainable [**2**], hereditarily indecomposable, and nearly homogeneous, and since each hereditarily indecomposable continuum has property K [**14**], it follows that the pseudocircle has uncountably many orbits.

Since each orbit of the pseudocircle under the action of the group G of homeomorphisms constructed in the second section meets uncountably many composants, it follows that each orbit of the pseudocircle under the action of $H(X)$ meets uncountably many composants.

Finally, in [**10**], Lewis proved the following theorem: Let V be an open subset of the pseudoarc P. Let p and q be distinct points of P such that the subcontinuum M, irreducible between p and q, does not intersect $\overline{V}$. Then there exists a homeomorphism k from P onto P with $h(p) = q$ and $k|V$ is the identity. Lewis has announced that the techniques of his proof apply as well to the pseudocircle; we use Theorem 7 to prove this directly.

Let p' and q' be points of the pseudocircle X such that p' and q' belong to the same composant of X. Let M' be the subcontinuum irreducible between p' and q'. Let $U = \{U_1, U_2, \ldots, U_n\}$ be a circular chain covering X such that neither $\overline{U}_1$ nor $\overline{U}_2$ contains a point of M'. Let $U = U_3 \cup \cdots \cup U_n$. Then $\{U_1, U_2, U\}$ is a three-linked circular chain covering X.

Use Theorem 7 to find an open set O of the pseudoarc such that $P - \overline{O} \neq \varnothing$ and a homeomorphism h from $\overline{O}$ to $\overline{U}$ such that $h(\partial\overline{O}) = \partial\overline{U}$. Let $p = h^{-1}(p')$ and $q = h^{-1}(q')$. Then $M = h^{-1}(M')$ is an irreducible continuum from p to q.

Let $k\colon P \to P$ be a homeomorphism such that $k(p) = q$ and $k|(P - O)$ is the identity. Then $h \circ k \circ h^{-1}$ is a homeomorphism of the pseudocircle onto itself that takes p' onto q'.

Hence each composant of the pseudocircle is a subset of some orbit of X, and the theorem is proved.

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